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Moving contact lines and rivulet instabilities. Part 1. The static rivulet

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A rivulet is a narrow stream of liquid located on a solid surface and sharing a curved interface with the surrounding gas. Capillary instabilities are investigated by a linearized stability theory. The formulation is for small, static rivulets whose contact (common or three-phase) lines (i) are fixed, (ii) move but have fixed contact angles or (iii) move but have contact angles smooth functions of contact-line speeds. The linearized stability equations are converted to a disturbance kinetic-energy balance showing that the disturbance response exactly satisfies a damped linear harmonic-oscillator equation. The 'damping coefficient' contains the bulk viscous dissipation, the effect of slip along the solid and all dynamic effects that arise in contact-line condition (iii). The 'spring constant', whose sign determines stability or instability in the system, incorporates the interfacial area changes and is identical in cases (ii) and (iii). Thus, for small disturbances changes in contact angle with contact-line speed constitute a purely dissipative process. All the above results are independent of slip model at the liquid–solid interface as long as a certain integral inequality holds. Finally, sufficient conditions for stability are obtained in all cases (i), (ii) and (iii).

1. Introduction

A rivulet is a narrow stream of liquid flowing along a solid surface and sharing an interface with the surrounding fluid. The flow within the rivulet is driven by the component of gravity along the solid surface as shown in figure 1. Rivulets are often seen on automobile windshields and on the walls of showers. They are frequently formed when uniform films break-up and during condensation processes. Although the rivulet flow-régime should strongly effect the heat/mass transport in these systems, no stability conditions or criteria are available.

Rivulets display a large variety of intriguing instability phenomena. Kern (1969, 1971) sees the break-up of rivulets into droplets, rivulet-meandering and the transition of rivulet flows from laminar to turbulent régimes. Large-amplitude surface waves are apparent in many situations. Our own preliminary experiments (Culkin 1979) show these plus other more intricate phenomena. Droplet formation is presumably due to capillary instability (Rayleigh 1879) where the surface tension on the liquid–air interface causes capillary pressure gradients that enhance small interfacial corrugations. Here, however, the liquid–solid interaction modifies the process. This mechanism is the object of the present study.

The feature of rivulets that makes them most interesting and also so difficult to

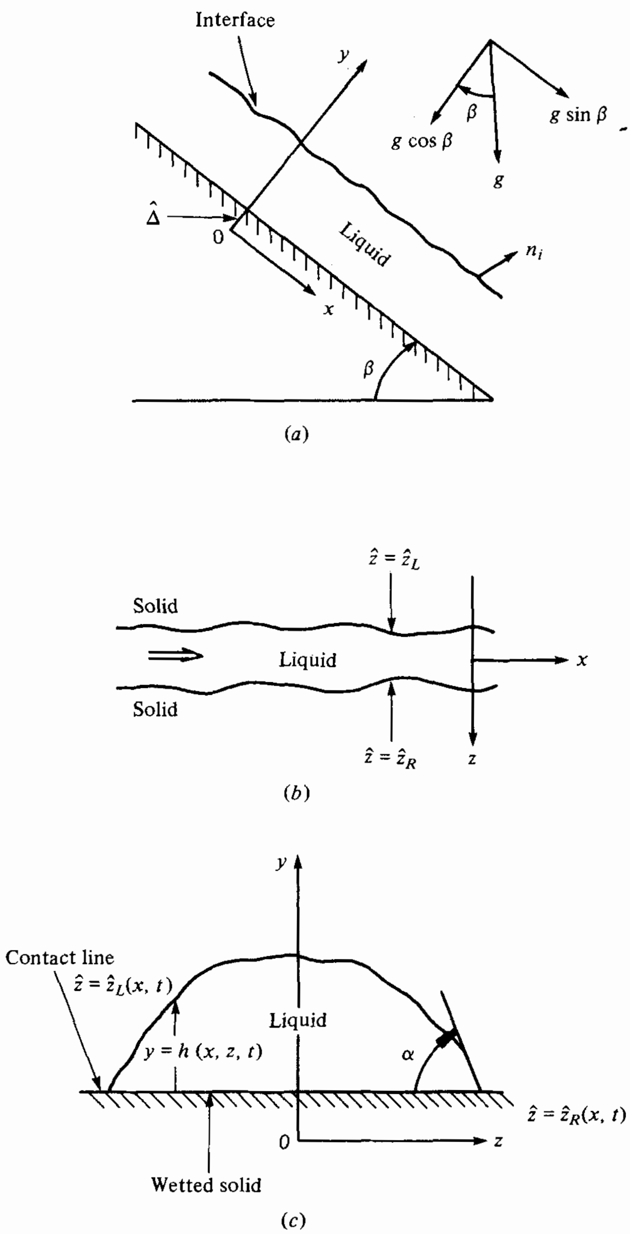

FIGURE 1. Sketch of a rivulet: (*a*) side view, (*b*) plan view, (*c*) front view.

analyse is the existence of their contact lines. A contact (or common or three-phase) line is a geometric curve formed when an interface between two immiscible fluids intersects a solid. The rivulet sketched in figure 1 shows two such lines. Even casual observation shows that owing to instability processes, the contact lines seem to move. The boundary conditions at moving contact-lines can be quite complicated. For example, Dussan V. & Davis (1974) have shown, using only the kinematics of the motion of moving contact-lines, that the usual no-slip condition applied everywhere on the wetted solid leads to a fluid dynamical model having infinite force near the contact line. If the local details of the flow are important, this is clearly unsatisfactory. An *effective slip condition* on the solid near the contact line will relieve the difficulty. This route has been used with success in several analyses involving mutual displacement of one viscous fluid by another (Dussan V. 1976; Hocking 1977; Huh & Mason 1977). These areas have been ably reviewed by Dussan V. (1979*a*).

The rivulet can thus be seen to have inherent fluid dynamical interest. It possesses free boundaries and moving contact lines, and it displays a large variety of instability phenomena. Furthermore, it can play the role of a vehicle for the study of moving-contact-line boundary conditions; by assessing their effects on the gross instability characteristics of the system, one might be able to infer those conditions appropriate to a given set of materials.

In the present work we begin such studies by focusing on the simplest possible rivulet. We examine a static rivulet (on a horizontal plate) that is so small that gravity effects can be ignored. There are two general methods for examining the stability of static states. On one hand one can use a *thermostatic approach* (Gibbs 1948); the state is stable if it is the minimum of an energy functional subject to appropriate constraints. The energy of the system consists of the gravitational potential, the interfacial energy on the fluid–fluid interface (surface tension times surface area) plus further energies, if any, attributable to the solid–fluid interface. The above defines a variational principal, the first variation of which yields the equilibria of the system. It is by consideration of the second variation that stability properties of such equilibria are obtained. Such analyses are often extremely complicated though certain cases of static rivulets have been analysed by Michael & Williams (1977).

The second approach to stability of static states is the usual *hydrodynamic theory*, say, for small disturbances. Here, the full dynamical system is analysed and conditions for stability or instability obtained. Dussan V. (1975) has shown using the hydrodynamic theory that in certain cases the thermostatic theory can be recovered and hence, the two can be equivalent. In certain cases the former is simpler to use while in other cases the latter can be simpler. However, when moving contact lines are present and the contact-line boundary conditions are dynamic in nature, then the thermostatic approach is inapplicable; the dynamic approach is the only alternative available.

In the present work we pose the linearized hydrodynamic stability theory for static rivulets. We consider several different cases of contact-line conditions. There are (i) fixed contact lines, (ii) fixed contact angles, (iii) contact angles that vary smoothly with contact-line speeds and (iv) contact angles that display hysteresis. In cases (ii), (iii) and (iv), slip between the liquid and the solid is allowed since in these cases the contact lines move when disturbed. The linear stability equations are then manipulated, incorporating the above contact-line conditions, to form a balance equation for the kinetic energy of the small disturbances. The result shows that small disturbances behave *exactly* as a *damped linear harmonic oscillator*. The disturbance kinetic energy plays the role of the mass and the effective viscous dissipation Φ_e plays the role of the damping coefficient. It is a combination I of changes in liquid–gas interfacial area and changes in fluid–solid interfacial area that plays the role of the spring constant. Stability is achieved when the coefficient I is positive. In case (i),

Φ_e is the viscous dissipation. In case (ii) Φ_e is the viscous dissipation modified by the slip behaviour on the solid; I is unaffected by the slip. In case (iii), which involves a dynamic contact line condition, Φ_e includes the dissipation, the slip along the solid plus the dynamic response of the contact line. The dynamic contact-line condition (iii) yields precisely the identical value of I that the fixed contact-angle condition (ii) does. Thus, the stability conditions for the two cases are identical. More importantly, this analysis shows that for small disturbances the dynamical effect of the contact angle varying with contact-line speed is a purely dissipative process. Finally, sufficient conditions for stability in all cases (i), (ii) and (iii) are obtained. Results are compared with the thermostatic analysis of Michael & Williams (1977). The case (iv) of contact-angle hysteresis is discussed and shown to be outside the realm of a linear stability theory.

2. Formulation

A long, smooth flat plate is inclined at an angle β to the horizontal. A narrow stream of liquid, a rivulet, flows down the plane as shown in figure 1. This Newtonian liquid has constant density ρ and constant viscosity μ. The flow is driven by the component $g \sin \beta$ of gravity along the plate and the system is isothermal. The surrounding fluid is a passive gas that applies a constant atmospheric pressure on the liquid–gas interface.

The governing equations for this system are the Navier–Stokes equations and the equation of continuity:

$$\rho(\hat{\mathbf{v}}_t + \hat{\mathbf{v}} . \nabla \hat{\mathbf{v}}) = \nabla . \hat{\boldsymbol{\sigma}} + \rho \hat{\mathbf{F}} \tag{2.1a}$$

and

$$\nabla . \hat{\mathbf{v}} = 0, \tag{2.1b}$$

where $\hat{\mathbf{v}}$ is the velocity vector $(\hat{u}, \hat{v}, \hat{w})$, $\hat{\boldsymbol{\sigma}}$ is the stress tensor,

$$\hat{\boldsymbol{\sigma}} = -\hat{\mathbf{p}}\mathbf{I} + \mu \hat{\mathscr{E}}, \tag{2.1c}$$

$$\mathscr{E} = \nabla \hat{\mathbf{v}} + (\nabla \hat{\mathbf{v}})^{\mathrm{T}}, \tag{2.1d}$$

and $\mathbf{F}$ is the body force per unit mass due to gravity,

$$\mathbf{F} = g(\sin \beta, -\cos \beta, 0). \tag{2.1e}$$

The superscript T denotes transpose.

Equations (2.1) are referred to a right-handed Cartesian co-ordinate system, shown in figure 1, whose x axis points down the plate and whose y axis points normal to the plate into the liquid. The origin of the coordinate system is defined in § 3; it is a distance $\hat{\Delta}$ from the plate.

The boundary conditions appropriate to the liquid–gas interface, at $\hat{y} = \hat{\eta}(\hat{x}, \hat{z}, \hat{t})$, are the kinematic condition,

$$\hat{v} = \hat{\eta}_{\hat{t}} + \hat{u}\hat{\eta}_{\hat{x}} + \hat{w}\hat{\eta}_{\hat{z}} \tag{2.2a}$$

and the stress jump appropriate to an uncontaminated interface having constant surface tension T,

$$[\![\hat{\sigma}_{ij}]\!] \hat{n}_j = 2T\hat{n}_i / R_m. \tag{2.2b}$$

Here $\hat{n}_j$ is the unit outward normal vector to the interface.

$$\hat{n}_i = (-\hat{\eta}_{\hat{x}}, 1, -\hat{\eta}_{\hat{z}})/(1 + \hat{\eta}_{\hat{x}}^2 + \hat{\eta}_{\hat{z}}^2)^{\frac{1}{2}}, \tag{2.2c}$$

and R_m is the mean radius of curvature of the surface,

$$2/R_m = [\hat{\eta}_{\hat{x}\hat{x}}(1+\hat{\eta}_{\hat{z}}^2) - 2\hat{\eta}_{\hat{x}}\hat{\eta}_{\hat{z}}\hat{\eta}_{\hat{x}\hat{z}} + \hat{\eta}_{\hat{z}\hat{z}}(1+\hat{\eta}_{\hat{x}}^2)]/(1+\hat{\eta}_{\hat{x}}^2+\hat{\eta}_{\hat{z}}^2)^{\frac{3}{2}}. \tag{2.2d}$$

The geometric curves of intersection between the interface and the plate are called *contact lines*. As shown in figure 1, these are located at $\hat{z} = \hat{z}_R(\hat{x},\hat{t})$ and $\hat{z} = \hat{z}_L(\hat{x},\hat{t})$. These positions are *a priori* unknown. (This is a free-boundary problem.) In order to complete the formulation of the flow problem, it is necessary to pose conditions on the motion of these lines.

The first statement is the *condition of contact*: There is a line along which the liquid thickness is zero,

$$\hat{\eta} = \hat{\Delta} \quad \text{at} \quad \hat{z} = \hat{z}_L, \hat{z}_R \quad \text{for all} \quad \hat{x}, \hat{t}, \tag{2.3a}$$

The second statement concerns the *contact angle* α; the slope of the interface at the contact line in the direction normal to the contact line is the tangent of α,

$$\nabla\hat{\eta}\,.\,\boldsymbol{\nu} = \mp \tan\alpha, \tag{2.3b}$$

where $\boldsymbol{\nu}$ is the unit outward vector *along the solid* and normal to the contact line. The $\pm$ refer to the pair of contact lines in question, respectively for + and − values $\hat{z} = \hat{z}_L$ and $\hat{z}_R$.

Before these two conditions can be converted into usable boundary conditions, they must be augmented by an *ansatz* that distinguishes one set of materials from another. This ansatz is ultimately dependent upon experimental observation. Among the possibilities are the following:

(i) *Fixed contact line*. The contact line does not move, its position remaining invariant for all time. Hence, $\hat{z}_L$ and $\hat{z}_R$ are time independent.

(ii) *Fixed contact angle*. The contact angle α does not differ from its static value α_0 for all time. Here it is presumed that α_0 is unique though this is a reasonable assumption in only few cases. See Dussan V. (1979*a*) for a discussion.

(iii) *Smooth contact-angle variation*. The contact angle α depends smoothly on the variables of the motion. For example, $\alpha = G(u_{CL})$ where u_{CL} is the speed of the contact line along the plate and $G'(u_{CL})$ exists always. The smoothness here *excludes* contact-angle hysteresis.

(iv) *Contact-angle hysteresis*. The contact angle depends on the motion but also on the history of the motion. For example, $\alpha = G(u_{CL})$ is discontinuous at $u_{CL} = 0$ as shown in figure 2. Such measurements are available for the apparent contact angle while it is widely believed that the actual contact angle α displays similar behaviour. See Dussan V. (1979*a*) for a discussion.

Finally, there is a boundary condition on the wetted solid. For the case (i) of a fixed contact line, there is the classical no-slip condition:

$$\hat{\mathbf{v}} = 0 \quad \text{on} \quad \hat{y} = \hat{\Delta}, \quad \hat{z}_L \leqslant \hat{z} \leqslant \hat{z}_R. \tag{2.4}$$

However, when the contact line can move as it would in cases (ii), (iii) and (iv), Dussan V. & Davis (1974) have shown on the basis of the kinematics that a non-integrable singularity at the contact line exists as long as the no-slip condition is enforced. Hence, we allow *effective slip* near each contact line. We shall use a slip model that gives the slip velocity as a linear function of the shear stress exerted by the liquid on the solid. This slippage applies in cases (ii), (iii) and (iv).

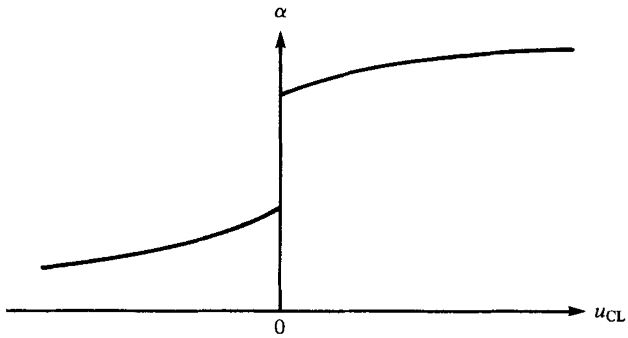

FIGURE 2. Sketch of experimental results of contact angle α *vs.* contact-line speed u_{CL}. $u_{CL} > 0$ denotes liquid displacing gas; $u_{CL} < 0$ denotes gas displacing liquid.

3. The basic static state

One solution (Towell & Rothfeld 1966) for rivulets on long smooth plates consists of a steady, fully developed flow that wets the plate on a strip of constant width,

$$-L \leqslant \hat{z} \leqslant L. \tag{3.1}$$

The corresponding interfacial shape is likewise x independent.

The *static basic state* we wish to consider is obtained by taking the $\beta \rightarrow 0$ limit of the Towell & Rothfeld solution. This rivulet on a horizontal plate is further considered so small that the Bond number $\rho g h_0^2/T \ll 1$ where h_0 is the maximum thickness of the rivulet. The resulting static (zero velocity) rivulet has a constant pressure and the interface takes the form of a cylindrical meniscus whose cross-section is the arc of a circle of radius R_0.

It is convenient to use cylindrical co-ordinates $(\hat{x}, \hat{r}, \hat{\theta})$ with the *origin at the centre of curvature of the meniscus*. The undisturbed interface then lies at $\hat{r} = R_0$. We scale as follows:

$$\text{length} \rightarrow R_0, \quad \text{speed} \rightarrow V_s = (T/\rho R_0)^{\frac{1}{2}}, \quad \text{pressure} \rightarrow T/R_0, \quad \text{time} \rightarrow R_0/V_s. \tag{3.2}$$

The cylindrical co-ordinates (x, r, θ) are given by $z = r\cos\theta$, $y = r\sin\theta$ as shown in figure 3 for a case when the static contact angle α_0 is less than $\frac{1}{2}\pi$.

In non-dimensional form, the basic state satisfies $\overline{p} \equiv 1$, $\hat{\mathbf{v}} \equiv 0$, and the meniscus given by

$$r \equiv 1, \quad \theta_0 \leqslant \theta \leqslant \pi - \theta_0. \tag{3.3}$$

Note that $\alpha_0 + \theta_0 = \frac{1}{2}\pi$.

4. The disturbance equations

Disturbances to the basic state are introduced as follows:

$$\mathbf{v} = 0 + \epsilon\mathbf{v}', \tag{4.1a}$$

$$\overline{p} = p + \epsilon p', \tag{4.1b}$$

where ϵ is a measure of disturbance amplitude. The interface lies at

$$r = 1 + \epsilon h' \tag{4.1c}$$

and the contact lines lie at

$$(z_L, z_R) = (-z_0, z_0) + \epsilon(-z'_L, z'_R), \tag{4.1d}$$

where $z_0 = \cos\theta_0$. The contact angle is given by

$$\alpha = \alpha_0 + \epsilon\alpha'$$

which corresponds to $\theta_{CL} = \theta_0 + \epsilon\theta'$ or $\pi - \theta_0 - \epsilon\theta'$.

If forms (4.1) are substituted into the non-dimensional form of (2.1)–(2.4), and linearized in disturbance quantities, the linearized disturbance equation result at order ϵ. These equations can then be expressed in terms of normal modes

$$\exp(\sigma_t + ikx). \tag{4.2}$$

The result is as follows:

$$\sigma \mathbf{v}' = \nabla . \boldsymbol{\sigma}', \tag{4.3a}$$

$$\nabla . \mathbf{v}' = 0, \tag{4.3b}$$

$$\mathbf{v}' . \mathbf{N} = \sigma h', \quad r = 1, \tag{4.3c}$$

$$\boldsymbol{\sigma}' . \mathbf{N} = \{(1-k^2)h' + h'_{\theta\theta}\}\mathbf{N}, \quad r = 1, \tag{4.3d}$$

$$v' = 0, \quad y = \Delta, \tag{4.3e}$$

$$\tau'_{2i} \equiv \mathbf{e}_i . \boldsymbol{\sigma}' . \mathbf{N} = \beta \mathbf{v} . \mathbf{e}_i, \quad i = 1, 3, \quad y = \Delta, \tag{4.3f}$$

where

$$\boldsymbol{\sigma}' = -p'\mathbf{I} + (\mu V_s/T)\mathscr{E}'. \tag{4.3g}$$

System (4.3) must be augmented by contact-line conditions. Here we have used the same symbols to denote the full, linearized unknown as its corresponding normal-mode amplitude. $\mathbf{N}$ is the unit outward normal to the basic-state interface and $\mathbf{e}_{(1)} = (1, 0, 0)$, $\mathbf{e}_{(2)} = (0, 1, 0)$, $\mathbf{e}_{(3)} = (0, 0, 1)$ are the Cartesian unit vectors.

Equation (4.3*a*) is the momentum balance and (4.3*b*) is the continuity equation. Forms (4.3*c*, *d*) are the kinematic and stress conditions on the interface which apply on the undisturbed position $r = 1$. Equations (4.3*e*, *f*) constitute the boundary conditions on the wetted plate located at $y = \Delta$. There is no cross-flow through the plate and along the plate the local slip velocity is proportional to the local shear stress. The slip coefficient is β. Within the linearization the directions in the plate tangent and normal to the contact line are respectively given by $e_i^{(1)}$ and $e_i^{(3)}$. Corrections in these directions are $O(\epsilon^2)$. Conditions (4.3*g*, *h*) are the conditions of contact and contact angle. Each apply on the undisturbed contact lines at $\theta = \theta_0$, $\pi - \theta_0$. The latter condition within the order of the linearization refers to the contact angle projected on the y, z plane. Again the error incurred in this simplification is $O(\epsilon^2)$.

5. Contact-line boundary conditions

In order to make the disturbance equations well-posed, boundary conditions at the contact lines are required.

In its disturbed state a rivulet has contact lines that are not straight lines but are corrugated curves that can move normal to themselves. It is in the normal directions

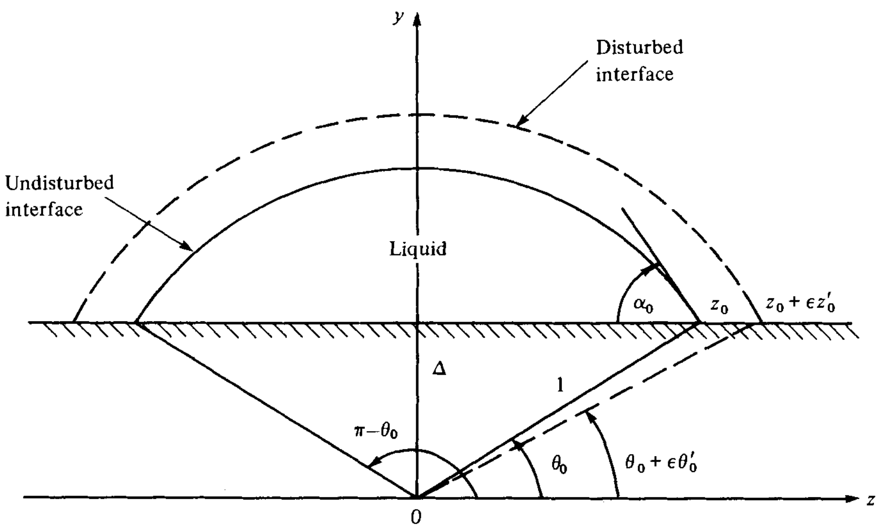

FIGURE 3. Sketch of cross-section of undisturbed and disturbed rivulet illustrating the cylindrical co-ordinate system (x, r, θ). The origin is the centre of curvature of the undisturbed meniscus.

that one defines planes that contain the contact angles. These normal directions can be taken as the z direction and the contact angles defined in the projected plane, the y, z plane. The error incurred is $O(\epsilon^2)$ and is therefore negligible according to linear stability theory. The projected plane is shown in figure 3.

In what follows we concentrate on the contact line near $z = z_0$, $y = \Delta$. (The results for the line near $z = -z_0$, $y = \Delta$ will merely be stated.) For convenience, we shall use a mixed notation, sometimes referring to (x, r, θ) and sometimes to (x, y, z). Hence, the contact line near $z = z_0$, $y = \Delta$ is equivalently near $r = 1$, $\theta = \theta_0$.

In order to define the interfacial slope (and hence the contact angle) at the contact line, we write a (Cartesian) position vector $\mathbf{R}$ to the disturbed interface at

$$r = h(x, \theta, t) = 1 + h'(x, \theta, t)$$

as follows: $\mathbf{R} = (x, h\sin\theta, h\cos\theta)$. A vector tangent to the interface (in the projected plane) is $\mathbf{R}_\theta = (0, h_\theta \sin\theta + h\cos\theta, h_\theta\cos\theta - h\sin\theta)$. The slope of the interface at the contact line defines the contact angle:

$$\left[\frac{h_\theta \sin\theta + h\cos\theta}{h_\theta\cos\theta - h\sin\theta}\right]_{\substack{y=\Delta \\ z=z_0+\epsilon z_0'}} = -\tan(\alpha_0 + \epsilon\alpha_0'). \tag{5.1a}$$

We also have two geometrical relations that are obvious from figure 3:

$$(z_0 + \epsilon z_0')^2 + \Delta^2 = (1 + \epsilon h')^2 \tag{5.1b}$$

and

$$\tan(\theta_0 + \epsilon\theta_0') = \frac{\Delta}{z_0 + \epsilon z_0'}. \tag{5.1c}$$

If relations (5.1) are expanded in powers of ϵ, we obtain at $O(1)$:

$$\cot\theta_0 = \tan\alpha_0, \tag{5.2a}$$

$$z_0^2 + \Delta^2 = 1, \tag{5.2b}$$

$$\tan\theta_0 = \Delta/z_0. \tag{5.2c}$$

These are the relationships appropriate in the basic state. Equations (5.1) give at $O(\epsilon)$ the following forms that apply at $y = \Delta$, $z = z_0$:

$$h'_\theta - \theta'_0 = \alpha'_0, \tag{5.3a}$$

$$z_0 z'_0 = h', \tag{5.3b}$$

$$\theta'_0 = -\left(\frac{\Delta}{z_0^2}\cos^2\theta_0\right) z'_0. \tag{5.3c}$$

Equations (5.2) and (5.3) can be manipulated to yield the desired relation for h':

$$h'_\theta + Sh' = \alpha'_0, \quad r = 1, \quad \theta = \theta_0 \tag{5.4a}$$

where

$$S = \cot\alpha_0. \tag{5.4b}$$

If a similar analysis to the above is applied to the contact line near $r = 1$, $\theta = \pi - \theta_0$, the result is as follows:

$$h'_\theta - Sh' = \alpha'_0, \quad r = 1, \quad \theta = \pi - \theta_0. \tag{5.5}$$

We can now turn to the cases outlined in § 2.

Case I. Fixed contact lines

The appropriate contact-line boundary condition results from the condition $z'_0 = 0$ and equation (5.3*b*),

$$h'(x, \theta_0, t) = 0 \quad \text{for all} \quad x, t \tag{5.6a}$$

and

$$h'(x, \pi - \theta_0, t) = 0 \quad \text{for all} \quad x, t. \tag{5.6b}$$

The normal-mode equivalent of these is clear. The boundary-value problem then gives h'_θ at each contact line and the dynamic contact angle is obtained through (5.3*a*) and (5.3*c*).

Case II. Fixed contact angle

The appropriate contact-line boundary condition results from condition $\alpha'_0 = 0$ and (5.4) and (5.5),

$$h'_\theta + Sh' = 0, \quad \theta = \theta_0 (r = 1) \quad \text{for all} \quad x, t \tag{5.7a}$$

$$h'_\theta - Sh' = 0, \quad \theta = \pi - \theta_0 (r = 1) \quad \text{for all} \quad x, t \tag{5.7b}$$

where

$$S = \cot\alpha_0. \tag{5.7c}$$

The normal-mode equivalent of these is clear. The boundary-value problem then gives h' from which (5.3*a*, *b*) determine z'_0 and α'_0 and hence the contact-line position and the contact angle.

Case III. Smooth contact-angle variation

If it is assumed that the instantaneous contact angle depends smoothly on the contact angle speed, then

$$\alpha_0 + \epsilon\alpha'_0 = G(0 + \epsilon\dot{z}'_0), \tag{5.8}$$

where the dot denotes the time derivative and the derivative G' exists and is continuous everywhere. By expanding (5.8) in powers of ϵ, we have at $O(1)$:

$$\alpha_0 = G(0). \tag{5.9}$$

Here α_0 is the (unique) static contact angle, the uniqueness resulting from the form (5.8) assumed. Equation (5.8) yields at $O(\epsilon)$:

$$\alpha'_0 = G_1 \dot{z}'_0, \tag{5.10a}$$

where

$$G_1 \equiv G'(0). \tag{5.10b}$$

We can combine (5.10) with (5.3*b*) to obtain

$$\begin{aligned}\alpha'_0 &= G_1 h'_t / z_0 \\ &= G_1 (\operatorname{cosec} \alpha_0) h'_t\end{aligned} \tag{5.11}$$

so that (5.4*a*) takes the form:

$$h'_\theta + Sh' = G_1 (\operatorname{cosec} \alpha_0) h'_t \quad \text{for all} \quad x, t. \tag{5.12}$$

Finally, if we introduce the normal-mode decomposition (4.2), the appropriate contact-line boundary conditions take the form

$$h'_\theta + (S - P\sigma) h' = 0, \quad \theta = \theta_0 (r = 1), \tag{5.13a}$$

$$h'_\theta - (S - P\sigma) h' = 0, \quad \theta = \pi - \theta_0 (r = 1), \tag{5.13b}$$

where

$$S = \cot \alpha_0 \tag{5.13c}$$

and

$$P = G_1 \operatorname{cosec} \alpha_0. \tag{5.13d}$$

The boundary-value problem determines h' and h'_θ at the contact line and the contact angle are determined through equation (5.11).

Case IV. Contact-angle hysteresis

Contact angle hysteresis is illustrated in figure 2 and corresponds to (5.8) where $G'(0)$ *does not exist*. Hence, where contact-angle hysteresis is present the functional G is inherently non-linearizable. If such a boundary condition as (5.8) were used, then a system having contact angle hysteresis could not be analysed using a linear stability theory.†

6. The energy equation

We shall obtain stability results from an energy-like integral form of the disturbance equations (4.3). To form this, we first define three integrals. A 'volume' integral of a quantity $Q(r, \theta)$ is denoted by $\int_{\mathscr{V}} Q$,

$$\int_{\mathscr{V}} Q = \int_{\theta=\theta_0}^{\pi-\theta_0} \int_{r=\Delta \operatorname{cosec} \theta_0}^{1} Q(r, \theta)\, r\, dr\, d\theta.$$

† See note added in proof.

An integral of a quantity $Q(r,\theta)$ is denoted by $\int_1 Q$,

$$\int_1 Q = \int_{\theta=\theta_0}^{\pi-\theta_0} Q(1,\theta)\,d\theta.$$

A 'surface' integral of a quantity $Q(r,\theta)$ is denoted by $\int_S Q$,

$$\int_S Q = \int_{z=-z_0}^{z_0} Q(r,\theta)|_{y=\Delta}\,dz.$$

The '*energy*' *integral* is obtained by taking the dot product of (4.3*a*) with the complex conjugate $\mathbf{v}'^*$ of $\mathbf{v}'$ and taking the volume integral. We obtain

$$\sigma\int_{\mathscr{V}} |\mathbf{v}'|^2 = \int_{\mathscr{V}} \mathbf{v}'^*.(\nabla.\boldsymbol{\sigma}') = \int_{\partial\mathscr{V}} \mathbf{v}'^*.\boldsymbol{\sigma}'.\mathbf{n} - \int_{\mathscr{V}} \nabla\mathbf{v}'^*:\boldsymbol{\sigma}, \tag{6.1}$$

where we have used the divergence theorem. The boundary $\partial\mathscr{V}$ of $\mathscr{V}$ consists of two parts. There is the interface $r=1, \theta_0 \leqslant \theta \leqslant \pi-\theta_0$, on which $\mathbf{n}=\mathbf{N}$ and (4.3*c*, *d*) apply. There is the wetted solid S at $y=\Delta$ on which $\mathbf{n}=-\mathbf{e}_{(2)}$ and (4.3*e*) apply. If these are used, equation (6.1) becomes

$$\sigma\int_{\mathscr{V}} |\mathbf{v}'|^2 = \int_1 \mathbf{v}'^*.\boldsymbol{\sigma}'.\mathbf{N} - \int_S \tau'_{21}u' - \int_S \tau_{23}w' - \int\nabla\mathbf{v}'^*:\boldsymbol{\sigma}'. \tag{6.2}$$

If we now use condition (4.3*f*) on $y=\Delta$, we can transform (6.2) into the following form

$$\sigma E + \Phi + I_1 = 0, \tag{6.3a}$$

where

$$E = \int_{\mathscr{V}} |\mathbf{v}'|^2 \tag{6.3b}$$

is the positive definite disturbance kinetic-energy and

$$\Phi = \int_{\mathscr{V}} \nabla\mathbf{v}'^*:\boldsymbol{\sigma}' + \int_S \beta[|u'|^2+|w'|^2] \tag{6.3c}$$

is the *effective* disturbance viscous dissipation. It is gratifying to see that effective slip of the liquid over the solid appears as a modification of the bulk dissipation. If $\beta>0$, Φ is positive definite. Even if β could be negative somewhere, one would expect the bulk integral to dominate over the surface integral ($|\beta|$ is very small). We shall assume† that $\Phi>0$. The interfacial integral I_1 is given by

$$I_1 = \int_1 \{(k^2-1)h' - h'_{\theta\theta}\}\mathbf{v}'^*.\mathbf{N}. \tag{6.3d}$$

On the interface, equation (4.3*c*), the kinematic condition allows us to transform (6.3) into the following form:

$$\sigma^2 E + \sigma\Phi + I = 0, \tag{6.4a}$$

where

$$I = \int_{\theta_0}^{\pi-\theta_0} \{(k^2-1)|\phi|^2 + |\phi_\theta|^2\}\,d\theta - [\phi^*\phi_\theta]_{\theta_0}^{\pi-\theta_0} \tag{6.4b}$$

† This assumption is no more than $\int_{\text{solid}} \mathbf{v}'.\boldsymbol{\sigma}'.\mathbf{n} \leqslant 0$, which is a condition that is independent of slip-model.

and

$$\phi \equiv \mathbf{v}'(1,\theta).\mathbf{N} \tag{6.4c}$$

is the interfacial normal speed. To obtain (6.4) we have used integration by parts on the term $\phi^*\phi_{\theta\theta}$. Equations (6.4) are the sought after form of the 'energy' balance.

7. Sufficient conditions for stability

The 'energy' balance equation, (6.4), allows us to pose a sufficient condition for stability.

Theorem: A static rivulet is stable (i.e. has only decaying normal modes) to infinitesimal disturbances when the function I, given in equation (6.4*b*), is positive.

This theorem is proved by solving the quadratic equation (6.4*a*) and examining the discriminant. The value of the functional I depends on the contact-line boundary conditions and so we must examine cases, one for each type of boundary condition posed.

Case I. Fixed contact lines

For fixed contact lines, equations (5.6) give

$$h' = 0, \quad \theta = \theta_0, \quad \pi-\theta_0$$

from which, together with (4.3*b*) and (6.4*c*), it follows that

$$\phi = 0, \quad \theta = \theta_0, \quad \pi-\theta_0. \tag{7.1}$$

Lemma 1. For all smooth enough functions ϕ that satisfy conditions (7.1), there exists a positive number ξ^2 such that

$$\int_{\theta_0}^{\pi-\theta_0} |\phi_\theta|^2 d\theta \geqslant \xi^2 \int_{\theta_0}^{\pi-\theta_0} |\phi|^2 d\theta. \tag{7.2a}$$

The largest such number is given by

$$\xi = \frac{\pi}{\pi-2\theta_0} = \frac{\pi}{2\alpha_0}. \tag{7.2b}$$

This is an elementary result from the calculus of variations (see Courant & Hilbert 1953, chap. IV). In forming (7.1*b*) we have used the result that $\alpha_0+\theta_0 = \frac{1}{2}\pi$, which follows from (5.2*a*).

Owing to conditions (7.1), the boundary terms in I vanish. Furthermore, the use of inequality (7.2) gives

$$I \geqslant (k^2+\xi^2-1)\int_{\theta_0}^{\pi-\theta_0} |\phi|^2 d\theta. \tag{7.3}$$

Hence, linearized stability is guaranteed if

$$k^2 \geqslant 1-\xi^2. \tag{7.4}$$

The range of possible instabilities vanishes if $1-\xi^2 < 0$ so that there is linearized stability for all k when

$$\alpha_0 < \tfrac{1}{2}\pi. \tag{7.5}$$

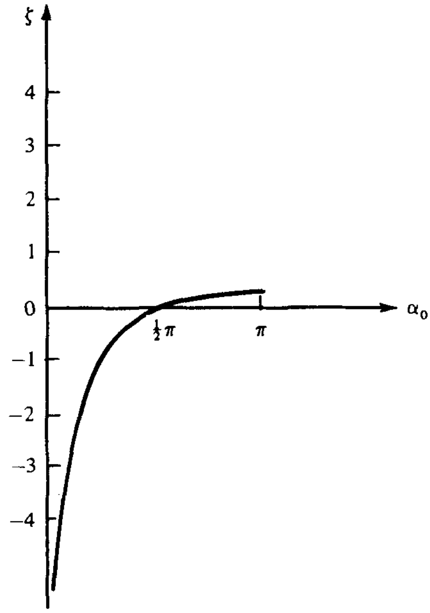

FIGURE 4. Calculated value of the isoperimetric constant ζ of system (7.7).

Case II. Fixed contact angles

Lemma 2. For all smooth enough functions ϕ, there exists a number ζ such that

$$\int_{\theta_0}^{\pi-\theta_0} |\phi_\theta|^2 \, d\theta - S\{|\phi(\theta_0)|^2 + |\phi(\pi-\theta_0)|^2\} \geqslant \zeta \int_{\theta_0}^{\pi-\theta_0} |\phi|^2 \, d\theta. \tag{7.6}$$

This inequality is obtained from a calculus of variations problem whose Euler–Lagrange equations are as follows:

$$\phi_{\theta\theta} + \zeta\phi = 0, \tag{7.7a}$$

$$\phi_\theta + S\phi = 0, \quad \theta = \theta_0, \tag{7.7b}$$

$$\phi_\theta - S\phi = 0, \quad \theta = \pi - \theta_0. \tag{7.7c}$$

with $S = \cot\alpha_0$. Figure 4 shows how the smallest ζ depends on α_0.

For fixed apparent contact angles, equations (5.7), (4.3*b*) and (6.4*c*) show that forms (7.7*b*) and (7.7*c*) apply. Using these, the functional I becomes

$$I = \int_{\theta_0}^{\pi-\theta_0} \{(k^2-1)|\phi|^2 + |\phi_\theta|\} \, d\theta - S\{|\phi(\pi-\theta_0)|^2 + |\phi(\theta_0)|^2\}. \tag{7.8}$$

If we use inequality (7.6), we obtain

$$I \geqslant (k^2+\zeta-1) \int_{\theta_0}^{\pi-\theta_0} |\phi|^2 \, d\theta. \tag{7.9}$$

Linearized stability follows for

$$k^2 > 1 - \zeta. \tag{7.10}$$

Case III. Smooth contact-angle variation

When the contact angle is a smooth function of contact-angle speed, the conditions (5.13) apply. In this case the boundary terms in I can be transformed as follows:

$$[\phi^*\phi_\theta]_{\theta_0}^{\pi-\theta_0} = (S-P\sigma)\{|\phi(\pi-\theta_0)|^2+|\phi(\theta_0)|^2\}. \tag{7.11}$$

If we now apply this result to (6.4a), we can redefine the effective dissipation Φ_e to be

$$\Phi_e = \Phi+P\{|\phi(\pi-\theta_0)|^2+|\phi(\theta_0)|^2\} \tag{7.12}$$

and the effective surface term I_e to be

$$I_e = \int_{\theta_0}^{\pi-\theta_0} \{(k^2-1)|\phi|^2+|\phi_\theta|^2\} - S\{|\phi(\pi-\theta_0)|^2+|\phi(\theta_0)|^2\}. \tag{7.13}$$

Here Φ_e is positive definite, the smooth contact-angle variation further modifying the 'viscous dissipation'. The new 'energy' equation has the new form

$$\sigma^2 E+\sigma\Phi_e+I_e = 0 \tag{7.14}$$

so that again stability is assured as long as

$$I_e > 0. \tag{7.15}$$

Condition (7.15) is precisely the same as that of case II, condition (7.10), so that the stability condition is the same as well. The modified dissipation merely affects the growth or decay rates of the modes, not the interval of stability.

Case IV. Contact-angle hysteresis

When contact-angle hysteresis is present, G of equation (5.8) is not smooth so that linearization is impossible. With the present formulation, equation (5.8), of this boundary condition only a nonlinear stability-theory is applicable.†

8. Discussion

Before we discuss the rivulet-stability results already obtained, let us reconsider the stability of a static, circular cylinder of liquid using our energy-integral technique. This capillary jet is equivalent to a rivulet with $\theta_0 = -\frac{1}{2}\pi$ or $\alpha_0 = \pi$ and *without liquid–solid contact*. In this case all dependent variables, and in particular ϕ, are 2π-periodic in θ so that the boundary terms in I sum to zero. Furthermore, since ϕ is 2π-periodic in θ, ϕ satisfies

$$\int_{-\frac{1}{2}\pi}^{\frac{3}{2}\pi} |\phi_\theta|^2\, d\theta \geqslant \xi^2 \int_{-\frac{1}{2}\pi}^{\frac{3}{2}\pi} |\phi|^2\, d\theta \tag{8.1a}$$

with

$$\xi = 0. \tag{8.1b}$$

Thus, we have for the *capillary jet*

$$I \geqslant (k^2-1)\int_{-\frac{1}{2}\pi}^{\frac{3}{2}\pi} |\phi|^2\, d\theta. \tag{8.2}$$

† See note added in proof.

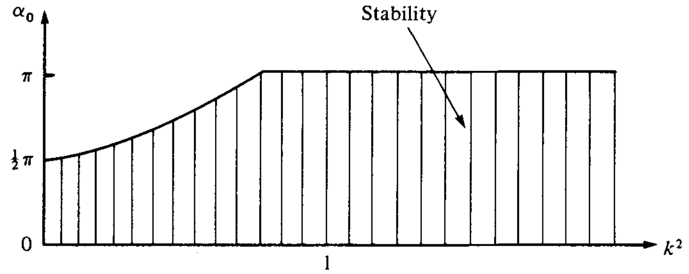

FIGURE 5. Predicted region (shaded) of stability, relation (7.4), for the static rivulet having fixed contact lines.

Linearized stability is thus assured if

$$k > 1. \tag{8.3}$$

Condition (8.3) is identical to the result of Rayleigh (1879), who derived this condition as both *necessary and sufficient for stability* when the liquid is inviscid. The only modes that can lead to instability are the axisymmetric ones. (This corresponds in our notation to $\xi = 0$ and the equality in form (8.1a).) Non-axisymmetric modes cannot grow. When the fluid is (Newtonian) viscous, the *same* ranges correspond to unstable ($0 < k < 1$) and stable ($k > 1$) modes (see e.g. Chandrasekhar 1961); the viscosity affects the magnitudes of the growth rates. We thus see that our bounding technique, when applied to the capillary jet, gives the *optimal range of stability*.

Let us now consider rivulets having *contact lines* that are *fixed*. We first examine a circular cylinder of viscous liquid whose only contact is with a thin, solid wire that touches the cylinder along a generator. In our notation this corresponds to $\theta_0 = -\frac{1}{2}\pi$ or $\alpha_0 = \pi$. Equations (7.2b) and (7.4) give a region of certain stability, $k^2 > \frac{3}{4}$, in contrast to the stability condition, $k^2 > 1$, for the case of the same cylinder without contact. The single fixed line of contact gives a substantial stabilization. In mathematical terms, the stabilization occurs because the fixed contact lines forbid there being a purely axisymmetric disturbance to the rivulet interface since all interfacial disturbances, to be admissible competitors, must satisfy the contact-line conditions.

Dussan V. (1979b) has examined this stability problem using an alternative minimum energy formulation (Dussan V. 1975) derived from the dynamics. She is able to obtain $k^2 > \frac{3}{4}$ as the *necessary and sufficient condition* for static stability. Again, our simple bounding technique gives the *optimal range of stability*.

We now return to the consideration of rivulets *per se*. When the *contact lines are fixed*, equations (7.2b) and (7.4) give a region of certain stability. Figure 5 shows this region as a function of the contact angle α_0. When $\alpha_0 < \frac{1}{2}\pi$, the rivulet is stable; when $\frac{1}{2}\pi < \alpha_0 < \pi$, the rivulet is stable for large enough k. There is a dramatic effect shown here due to the liquid–solid contact and the conditions of fixed contact lines. When $\alpha_0 < \frac{1}{2}\pi$, the rivulet is *unconditionally stable* to small disturbances. The corresponding capillary jet is *unconditionally unstable* (since general disturbances must contain wavenumbers in the range (0, 1)). We can also consider the case $\alpha_0 = \pi$ which corresponds to a full, circular cylinder of liquid placed on the solid plate with only a single line, a generator, touching the solid. Our results here, that there is stability for $k^2 > \frac{3}{4}$, is identical with that for the case discussed above in which the cylinder contacts a wire. In that case $k^2 > \frac{3}{4}$ was both necessary and sufficient for stability. Here, the

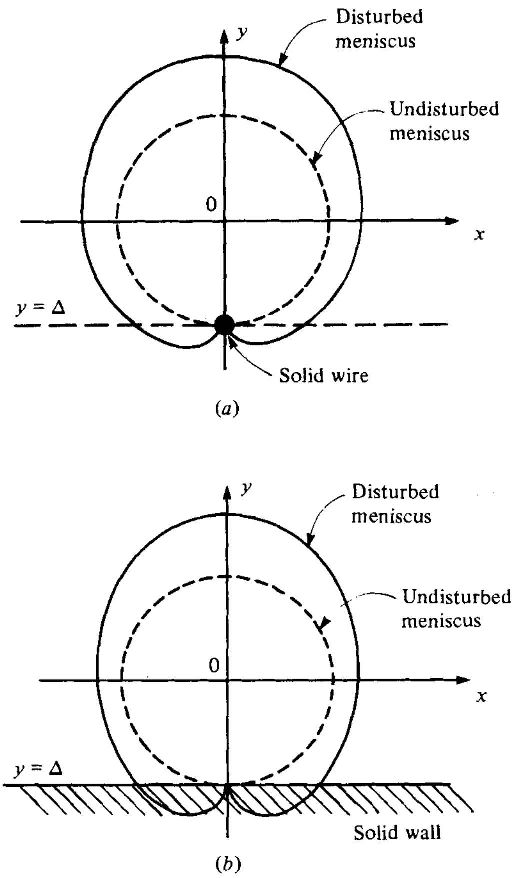

FIGURE 6. Sketch of the undisturbed and disturbed meniscus for the case of $\alpha_0 = \pi$ and a fixed contact line (*a*) for liquid contacting a wire, (*b*) for liquid contacting a plate. The above interfacial expansion due to disturbance occurs for half of each wavelength $2\pi/k$ while in the other half wavelength the interface contracts.

condition is certainly sufficient but may not be necessary.† To see this we note that the minimizing function ϕ_M of relation (7.2*a*) is $\phi_M = \sin \frac{1}{2}(\theta + \frac{1}{2}\pi)$. The corresponding admissible interfacial-position disturbance is proportional to ϕ_M through equation (4.3*c*). It is easy to see by plotting the distorted interface shape that the maximizing function corresponds to the interface crossing the line $y = \Delta$. As shown in figure 6(*a*) for the wire case, this is dynamically allowable. However, as shown in figure 6(*b*) in the plate case, this corresponds to liquid mass penetrating the solid plate. Since this is not *dynamically* allowable, the sufficient condition $k^2 > \frac{3}{4}$ for stability is probably not a necessary condition as well.

All of the above cases involve fixed contact lines so that they are in the realm of the thermostatic theory of Michael & Williams (1977). They consider a rivulet on a solid strip of fixed width $2L$ whose contact lines are fixed on the corners. They find for their fixed-volume case instabilities of the type discussed here but it is difficult to interpret their results in terms of the present parameters. (They scale lengths on the capillary

† This observation and the subsequent argument are due to Prof. E. B. Dussan V.

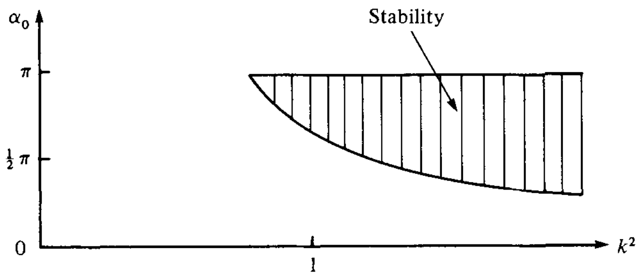

FIGURE 7. Predicted region (shaded) of stability, relation (7.10), for the static rivulet having either fixed contact angle or contact angle a smooth function of contact-line speed.

length $(T/\rho g)^{\frac{1}{2}}$ so that it is difficult to take the limit $g \to 0$ and they do not present their results in terms of contact angle.)

When the *contact angle is fixed*, equation (7.10) and figure 4 give a region of certain stability. Figure 7 shows this region as a function of the contact angle α_0. For all α_0 the region of certain stability fails to encompass all possible axial wavenumbers k; there is always a non-zero cut-off value. Again, the case $\alpha_0 = \pi$ provides an interesting example of liquid-solid-interaction effects. We see that in this case, a circular capillary jet making contact with the solid along one generator, is stable in a range $k < 1$ where the capillary jet would be unstable. Michael & Williams (1977) claim in this case that their thermostatic theory for the fixed-volume case would give stability for all α_0.

When there is *smooth contact-angle variation*, the thermostatic theory is inapplicable. Our bounding technique gives the *same range* of certain stability as that in figure 7. All variations in the contact-angle lead to modifications in the effective viscous dissipation and hence to modifications in the growth rates.

No results have been obtained for a system that displays *contact-angle hysteresis*. According to the formulation in equation (5.8), this phenomenon, although physically common, is outside the realm of linear stability theory.†

9. Conclusions

In the present work we have formulated the linearized stability theory of small, static rivulets. Three types of contact-line conditions have been defined, the conditions being posed in terms of the perturbation h' in the free-surface position. (i) When a *contact line is fixed*, $h' = 0$. (ii) When a *contact angle is fixed*, there is a mixed condition (5.7) in which a linear combination of h' and its spatial derivative vanishes. (iii) When there is smooth variation of the contact angle, there is a time-dependent condition (5.12) in which a linear combination of h', its spatial derivative and its time derivative vanishes.

The linearized stability equations and boundary conditions are converted into a balance equation for kinetic energy. The disturbance response is given exactly by a *damped linear harmonic-oscillator* whose disturbance kinetic energy corresponds to the mass, whose effective dissipation Φ_e corresponds to the damping coefficient and whose interfacial area changes I corresponds to the spring constant. Φ_e contains the bulk viscous dissipation, the effect of slip (in cases (ii) and (iii)) and the dynamic

† See note added in proof.

response of the contact-angle variations with speed (case (iii)). The result is *independent of slip model* as long as

$$\int_{\text{solid}} \mathbf{v}' . \boldsymbol{\sigma}' . \mathbf{n} \leqslant 0.$$

This formulation shows that smooth variations in contact angle with contact-line speed constitutes a purely dissipative process since this effect contributes to Φ_e only while leaving I unchanged. The sign of I determines stability ($I > 0$) or instability ($I < 0$).

Sufficient conditions for stability are obtained in various cases of (i), (ii) and (iii) using bounding techniques. These offer easily accessible results in cases (i) and (ii) when a thermostatic theory would apply, as well as in case (iii) when the thermostatic formulation is inapplicable. Since I for cases (ii) and (iii) are identical, the stability criteria in these two cases are identical. These stability results in terms of the contact-line conditions are a first step toward a general understanding of the effects of contact-line dynamics upon the gross stability behaviour of interfacial systems.

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Note added in proof (2 January 1980)

If one thinks of the relation $\alpha = G(u_{CL})$ of figure 2 as being represented as the limit of a sequence of steeper and steeper smooth curves having $G'(0) \to \infty$, then the stability result of case III would apply to the contact-angle hysteresis case IV as well. No direct analysis of this case exists, though.

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